1 Remarks

(i) $I, J \subset \mathfrak{g}$ ideals $\quad I+J := \{x+y \mid x \in I, y \in J\}$

$$[x+y, x'+y'] = \underbrace{[x,x']}_{\in I} + \underbrace{[x,y'] + [y,x']}_{\in I \cap J} + \underbrace{[y,y']}_{\in J} \qquad (1)$$

$- ad_{y'}(x) + ad_y(x')$

Hence it is a Lie sub-algebra.

In fact $[x+y, z] = \underbrace{[x,z]}_{\in I} + \underbrace{[y,z]}_{\in J}, \quad \in I+J$

$$[x+y, z] = \underbrace{[x,z]}_{\in I} + \underbrace{[y,z]}_{\in J} \in I+J$$

$I+J$ is still an ideal.

(ii) I an ideal $\quad J$ an sub-algebra

$$I+J = \{x+y \mid x \in I, y \in J\}$$

By (1) it is an algebra. $\qquad$ Not an ideal.

Levi decomposition is $\quad \mathfrak{g} = Rad(\mathfrak{g}) + \eta \quad$ η is a sub-algebra

$$Rad(\mathfrak{g}) \cap \eta = \{0\}.$$

As a linear space it is a direct sum.

But as Lie-algebras, it is so-called a semi-direct product

$$\mathfrak{g} = \eta \oplus Rad(\mathfrak{g})$$

η: Semi-Simple $\qquad Rad(\mathfrak{g})$: Soluble

Since $\mathfrak{g}/Rad(\mathfrak{g}) \simeq \eta$ as Lie algebra

& $\mathfrak{g}/Rad(\mathfrak{g})$ is semi-simple. (otherwise

From $0 \to Rad(\mathfrak{g}) \longrightarrow \mathfrak{g} \xrightarrow{\phi} \mathfrak{g}/Rad(\mathfrak{g}) \longrightarrow 0$

$$\mathfrak{g}/Rad(\mathfrak{g}) \supseteq C \qquad C \text{ Abelian}$$

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \phi^{-1}(\mathfrak{c}) \xrightarrow{\phi} \mathfrak{c} \to 0 \quad \Longrightarrow \quad \begin{array}{c} \phi^{-1}(\mathfrak{c}) \text{ is Solvable} \\ \cup \\ \mathrm{Rad}(\mathfrak{g}) \end{array}$$

Hence $\mathfrak{h}$ in the Levi-decomposition is always Semi-simple.

(iii). $B|_{\mathfrak{g}'} \equiv 0 \overset{?}{\Longleftrightarrow}$ $\boxed{\mathrm{tr}(ad_x \circ ad_y) = 0 \quad \forall\, x \in \mathfrak{g}' = [\mathfrak{g}\,\mathfrak{g}],\ y \in \mathfrak{g}}$ //

Since $\displaystyle\sum_{i=1}^{n} \langle ad_x \circ ad_y(e_i),\ e_i^* \rangle$

$$= \Big(\underbrace{\sum_{i=1}^{k} \langle ad_x \circ ad_y(e_i),\ e_i^* \rangle}_{} \Big) \quad (=\text{RHS})$$

$\{e_i\}_{i=1}^{k}$ a basis of $\mathfrak{g}'$ extends into $\{e_i\}_{i=1}^{n}$

$B|_{\mathfrak{g}'} \equiv 0 \Longleftrightarrow$ $\left[\forall x, y \in \mathfrak{g}'\ \ \text{RHS} = 0. \right]$

Hence it is a bit weaker than $\mathrm{tr}(ad_x \cdot ad_y) = 0,\ \forall\, x \in \mathfrak{g}',\ y \in \mathfrak{g}$

However if $B|_{\mathfrak{g}'} \equiv 0 \Rightarrow \mathfrak{g}'$ is Solvable by Cartan's Criterion [Not Ziller version]

$\Rightarrow \mathfrak{g}$ is solvable since $\mathfrak{g}/\mathfrak{g}'$ is Abelian. Hence Solvable.

Namely, Ziller's Statement of Cartan's Criterion is ok.

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(2) Examples

(i) $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. $\quad A \quad \mathrm{tr}(A) = 0$

Clearly $\overline{A}^t \in \mathfrak{g}$ as well, if $A \in \mathfrak{g}$

$$B(X,Y) = 2n\,tr(XY) - \underline{2\,tr(X)}\cdot\underline{tr(Y)} \quad \text{on } \mathfrak{gl}(n,\mathbb{C})$$

$$\Rightarrow \quad B_{\mathfrak{g}} = B|_{\mathfrak{g}} \quad \Rightarrow \text{For, } A, \bar{A}^{tr} \in \mathfrak{g}$$

$$B_{\mathfrak{g}}(A, \underbrace{\bar{A}^{tr}}_{\in\, \mathfrak{sl}(n,\mathbb{C}) = \{A \mid tr(A)=0\}}) = 2n\,tr(A\bar{A}^{tr}) = 2n \sum_{i,j} a_{ij}\bar{a}_{ij} > 0$$

if $A \neq 0$

Hence B is Not degenerate.

(ii) Similarly $\mathfrak{sl}(n,\mathbb{R})$ is semi-simple. $\rightarrow$ $A \in \mathfrak{sl}(n,\mathbb{R})$, $A^{tr} \in \mathfrak{sl}(n,\mathbb{R})$

$$u(n) := \{ A \mid A + \bar{A}^{tr} = 0 \}, \quad \bar{A}^{tr} + \overline{\bar{A}^{tr}}^{\,tr} = 0$$

$u(n)$ is not semi-simple: $X = \underset{\sim}{Fiid} \in u(n)$

$ad_X = 0$ $\quad ad_X(Y) = \underbrace{XY - YX}_{=0}$

Not an ideal of $\mathfrak{gl}(n)$

$[A,B] \subsetneq u(n)$

$A \in u(n)$

$B \in \mathfrak{gl}(n)$

needs its own calculation

$n \geq 2$ it is

(iii) $\mathfrak{g} := \text{Span}\{X, Y\}$ $\quad [X,Y] = Y$

$$\Rightarrow \quad Rad(\mathfrak{g}) = \mathfrak{g}.$$

$$B(X,X) = \langle ad_X\, ad_X(Y), Y^* \rangle = 1$$

$$B(X,Y) = \langle ad_X\, \underbrace{ad_Y(X)}_{0}, X^* \rangle + \langle ad_X\, \underbrace{ad_Y(Y)}_{0}, Y^* \rangle = 0$$

$$B(Y,Y) = \langle ad_Y\, ad_Y(X), \underbrace{X^*}_{0} \rangle + \langle ad_Y\, \underbrace{ad_Y(Y)}_{0}, Y^* \rangle$$

$Rad(B) = \text{Span}\{Y\}$. $\qquad Rad(B) \neq Rad(\mathfrak{g})$.

Is $Rad(B) \subset Rad(\mathfrak{g})$?

Yes. ~~Kevin Dao will show us.~~ Homework

③ Structure theorem for $\mathfrak{g}$ – a Semi-simple Lie algebra

ⓐ $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$

each $\mathfrak{g}_i$ is simple

ⓑ $\forall\, \mathfrak{a}$ an ideal of $\mathfrak{g}$ $\quad \mathfrak{a} = \mathfrak{g}_{i_1} \oplus \cdots \oplus \mathfrak{g}_{i_s}$

ⓒ $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$

ⓓ $\mathrm{Aut}(\mathfrak{g}) / \mathrm{Int}(\mathfrak{g})$ is discrete.

Def. $\mathfrak{a} \subset \mathfrak{g}$ an ideal is called simple, if (i) $\mathfrak{a}$ is Not abelian namely $[\mathfrak{a}, \mathfrak{a}] \neq 0$
(ii) $\mathfrak{a}$ contains no nontrivial ideal, (except 0, & $\mathfrak{a}$).

$\Longrightarrow [\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$ if $\mathfrak{a}$ is simple.

Simple ideal is Semi-simple. Since $\mathrm{Rad}(\mathfrak{a}) = \{0\}$.
is an ideal.

Ⓐ
Key lemma, $\mathfrak{a} \subset \mathfrak{g}$ is an ideal $\quad \beta$ – invariant bilinear form.
$\beta|_{\mathfrak{a}}$ is non-degenerate. $\quad \mathfrak{a}^{\perp} := \{ x \in \mathfrak{g} \mid \beta(x, y) = 0, \ \forall y \in \mathfrak{a} \}$

Then $\mathfrak{a}^{\perp}$ is an ideal, & $\quad \mathfrak{a} \cap \mathfrak{a}^{\perp}$ is Abelian.

pf. (i) $\quad x \in \mathfrak{a}^{\perp}, \quad y \in \mathfrak{g} \quad z \in \mathfrak{a}$

$$\beta([x, y], z) = \beta(x, \underbrace{[y, z]}_{\in \mathfrak{a}}) = 0 \quad \Longrightarrow \quad [x, y] \in \mathfrak{a}^{\perp}$$

(ii) $\forall\, x, y \in \mathfrak{a} \cap \mathfrak{a}^{\perp} \ (\subset \mathfrak{a})$

$$\beta([x, y], z) = \beta(x, \underbrace{[y, z]}_{\in \mathfrak{a}}) = 0 \quad \Longrightarrow \quad [x, y] = 0.$$

The same argument shows if β is non-degenerate on $\mathfrak{a}$, $[x, y] = 0$ as well.

(B) $\dim(\mathfrak{a}) + \dim(\mathfrak{a}^\perp) = \dim(\mathfrak{g})$. Let $\{e_i\}_{i=1}^{k}$ basis of $\mathfrak{a}$.

$x = \sum x^i e_i$ $\quad x \in \mathfrak{a}^\perp$

$$x^i \sum_{i=1}^{n} \underbrace{\beta(e_i, e_j)}_{\beta_{ij}} = 0 \quad \forall \quad 1 \le j \le k \qquad (*)$$

$$(\beta_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le k}} \quad \text{has rank } k$$

Otherwise $\exists \lambda^j$ $(\lambda^1 \dots \lambda^k) \neq 0$ such that $\sum_{j=1}^{k} \beta_{ij} \lambda^j = 0$

$$\Rightarrow \quad \beta(e_i \; \sum_{j=1}^{k} \lambda^j e_j) = 0 \quad \forall i$$

$$\Rightarrow \quad \sum_{j=1}^{n} \lambda^j e_j \in \mathrm{Rad}(\beta).$$

Linear algebra $\Rightarrow$ $\dim(\mathfrak{a}^\perp) = \dim(\text{Solution space of } (*))$

$= n - k$. $\square$

Then (A) + (B), if $\mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$, which is the case if B is non-degenerate ($\mathfrak{g}$ is semi-simple, and Abelian ideal must be zero),

$$\Rightarrow \quad \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp \quad (\mathfrak{a} \times \mathfrak{a}^\perp)$$

Applying the procedure repeatedly $\Longrightarrow$

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$$

Note also, when $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, $\quad B_\mathfrak{g} = B_\mathfrak{a} + B_{\mathfrak{a}^\perp}$

$$B_\mathfrak{g}(x, y) = B(x_1 + x_2, y_1 + y_2) = B(x_1, y_1) + B(x_2, y_2)$$
$$= B_\mathfrak{a}(x_1, y_1) + B_{\mathfrak{a}^\perp}(x_2, y_2).$$

This proves (a). $\Rightarrow$ $B_\mathfrak{a}$ is non-degenerate since if $B_\mathfrak{a}(x, y) = 0$ $\forall y \in \mathfrak{a}$ $\Rightarrow$ $B(x, z) = 0$ $\forall z$ "$y + y^\perp$" since $B(x, y^\perp) = 0$, if $x \in \mathfrak{a}$ & $y^\perp \in \mathfrak{a}^\perp$

For (b), it is clearly $\mathfrak{a} \cap \mathfrak{g}_i$ is either $= \mathfrak{g}_i$ or 0. ($\mathfrak{g}_i$ is simple).

Hence $\quad \mathfrak{a} = \mathfrak{g}_{i_1} \oplus \cdots \oplus \mathfrak{g}_{i_s}$

For (c). simply note for $\mathfrak{g}_i$ $\quad [\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ due to $\mathfrak{g}_i$ is simple.

For (d): $\quad A \in \mathrm{Der}(\mathfrak{g})$. $\Longrightarrow$ $A([X, Y]) = [AX, Y] + [X, AY]$

$$\Longrightarrow \quad A \cdot \mathrm{ad}_X = \mathrm{ad}_{AX} + \mathrm{ad}_X A$$

Namely $\quad A \cdot \mathrm{ad}_X - \mathrm{ad}_X A = \mathrm{ad}_{AX}$

$\left\{ \begin{array}{l} \mathrm{int}(\mathfrak{g}) \\ := \mathrm{ad}_\mathfrak{g} \end{array} \right.$

[which shows $\mathrm{int}(\mathfrak{g}) := \mathrm{ad}_\mathfrak{g}$ is an ideal]

$\boxed{\mathrm{Int}(\mathfrak{g})}$ = Lie group with Lie algebra $\mathrm{int}(\mathfrak{g})$.

$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{int}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is an isomorphism

since $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

$\Longrightarrow$ $\mathrm{int}(\mathfrak{g})$ is semi-simple

$\Longrightarrow$ $\mathrm{int}(\mathfrak{g}) \cap \mathrm{int}(\mathfrak{g})^\perp = \{0\}$ in $\mathrm{Der}(\mathfrak{g})$.

by the Key Lemma it is Abelian $= 0$ due to $\mathrm{int}(\mathfrak{g})$ is semi-simple.

$\Rightarrow$ If $A \in \mathrm{int}(g)^\perp$

$$[A, \mathrm{ad}_x] = \mathrm{ad}_{Ax} = 0$$

On $\mathrm{int}(g)$ B is non degenerate.

$[A, \mathrm{ad}_x] = 0$ by $\in \mathrm{int}(g) \cap \mathrm{int}(g)^\perp$

$Ax \in \mathfrak{z}(g)$

$\Rightarrow$ $Ax = 0$ $\forall x$ $\Rightarrow$ $A = 0$. $\square$.

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④ Furthe remarks.

(i) $g = g_1 \oplus g_2$ we mean it is both as Lie algebra & as vector space

$$\big[(u, v), (u', v')\big] = \big([u, u'], [v, v']\big)$$

In general $[u+v, u'+v'] = [u\ u'] + [v, v'] + \underbrace{[u\ v'] + [v, u']}_{\in g_1 \cap g_2 = \{0\}}$.

(ii) g_i is simple $\Rightarrow$

$\phi(g_i)$ contains no non trivial ideals.

$0 \to \ker(\phi) \to g \xrightarrow{\phi} \phi(g) \to 0$

$0 \neq a \subsetneq \phi(g)$ $\quad \phi^{-1}(a) \subsetneq g$.

Hence if $x \in \phi^{-1}(a)$, $y \in g$

$\phi([x, y]) = [\phi(x), \phi(y)] \in a$, $\phi(x) \in a$, if a is an ideal of $\phi(g)$

$\Rightarrow [x\ y] \in \phi^{-1}(a)$.

Hence image of a simple algebra has no nontrivial ideals

If $\phi(g)$ is 1-dimensional, or $[\phi(g), \phi(g)] = 0$

$[\phi(g), \phi(g)] = \phi([g, g]) = \phi(g)$

$\Rightarrow \phi(g) = 0$. Hence If $\phi(g) \neq 0 \Rightarrow \phi(g)$ is simple.

Hence we have $\phi(g)$ is simple.

Then if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$

$\phi(\mathfrak{g}) = \phi(\mathfrak{g}_1) \oplus \cdots \oplus \phi(\mathfrak{g}_k)$. is semi-simple by below.

Namely the image of semi-simple Lie algebra is semi-simple.

Claim if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ $\mathfrak{g}_i$ simple

$\Rightarrow$ $\mathfrak{g}$ is semi-simple.

W.L.G we consider it over $\mathbb{C}$.

$B = B_1 \oplus \cdots \oplus B_k$

$B(x, y) = 0$ $\forall y \Rightarrow$ $B_i(x_i, y_i) = 0$ $\forall y_i \in \mathfrak{g}_i$ $\Rightarrow x_i = 0$ $\Big\}$ $\Rightarrow$ $x = 0$

$x = \sum x_i$.

$y = \sum y_i$.